

Algebraic relations for reciprocal sums of even terms in Fibonacci numbers

CARSTEN ELSNER carsten.elsner@fhdw.de
*Fachhochschule für die Wirtschaft, University of Applied Sciences,
 Freundallee 15, D-30173 Hannover, Germany*

SHUN SHIMOMURA shimomur@math.keio.ac.jp
 IEKATA SHIOKAWA shiokawa@beige.ocn.ne.jp
*Department of Mathematics, Keio University,
 3-14-1 Hiyoshi, Kohoku-ku, Yokohama 223-8522 Japan*

Let $\{F_n\}_{n \geq 0}$ and $\{L_n\}_{n \geq 0}$ be Fibonacci numbers and Lucas numbers defined by

$$\begin{aligned} F_0 = 0, \quad F_1 = 1, \quad F_{n+2} &= F_{n+1} + F_n \quad (n \geq 0), \\ L_0 = 2, \quad L_1 = 1, \quad L_{n+2} &= L_{n+1} + L_n \quad (n \geq 0). \end{aligned}$$

Duverney, Ke. Nishioka, Ku. Nishioka, and the last named author [1] proved the transcendence of the numbers

$$\sum_{n=1}^{\infty} \frac{1}{F_n^{2s}}, \quad \sum_{n=1}^{\infty} \frac{1}{L_n^{2s}}, \quad \sum_{n=1}^{\infty} \frac{1}{F_{2n-1}^s}, \quad \sum_{n=1}^{\infty} \frac{1}{L_{2n}^s} \quad (s = 1, 2, 3, \dots)$$

by using Nesterenko's theorem ([3]) on Ramanujan functions $P(q)$, $Q(q)$, and $R(q)$.

Let us set

$$\zeta_F(s) := \sum_{n=1}^{\infty} \frac{1}{F_n^s} \quad (s = 1, 2, 3, \dots).$$

In [2], we proved that the values $\zeta_F(2)$, $\zeta_F(4)$, $\zeta_F(6)$ are algebraically independent, and that for any integer $s \geq 4$,

$$\zeta_F(2s) - 5^{s-2} r_s \zeta_F(4) \in \mathbb{Q}(u, v), \quad u = \zeta_F(2), \quad v = \zeta_F(6)$$

with some $r_s \in \mathbb{Q}$ ($r_s = 0$ if and only if s is odd), where the rational function of u and v is explicit; for example,

$$\begin{aligned} \zeta_F(8) - \frac{15}{14} \zeta_F(4) &= \frac{1}{378(4u+5)^2} \left(256u^6 - 3456u^5 + 2880u^4 + 1792u^3v \right. \\ &\quad \left. - 11100u^3 + 20160u^2v - 10125u^2 + 7560uv + 3136v^2 - 1050v \right). \end{aligned}$$

Similar results were obtained for $\sum_{n=1}^{\infty} (-1)^{n+1} F_n^{-2s}$, $\sum_{n=1}^{\infty} L_n^{-2s}$, and $\sum_{n=1}^{\infty} (-1)^{n+1} L_n^{-2s}$.

In this talk, we discuss the algebraic independence and algebraic relations for reciprocal sums of even terms including

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n}^{2s}}, \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{F_{2n}^{2s}}, \quad \sum_{n=1}^{\infty} \frac{1}{L_{2n}^p}, \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{L_{2n}^p}.$$

Furthermore, combining two different even sums, we obtain reciprocal sums of evenly even terms and of unevenly even terms. For example, from $\sum_{n=1}^{\infty} F_{2n}^{-2s}$ and $\sum_{n=1}^{\infty} (-1)^{n+1} F_{2n}^{-2s}$, we derive

$$\Theta(2s) = \sum_{n=1}^{\infty} \frac{1}{F_{4n}^{2s}}, \quad \Theta^{\#}(2s) = \sum_{n=1}^{\infty} \frac{1}{F_{4n-2}^{2s}}.$$

We show that the numbers $\Theta(2)$, $\Theta(4)$, $\Theta(6)$ (respectively, $\Theta^{\#}(2)$, $\Theta^{\#}(4)$, $\Theta^{\#}(6)$) are algebraically independent, and for each $s \geq 4$, $\Theta(2s)$ (respectively, $\Theta^{\#}(2s)$) is written as an algebraic function of them. In particular, the algebraic relations corresponding to $\Theta^{\#}(2s)$ are rational; for example,

$$\Theta^{\#}(8) = \frac{1}{70(5x + 6y)} \left(125x^3 + 450x^2y + 540xy^2 + 216y^3 \right. \\ \left. - 125xy - 500xz + 50y^2 - 200yz + 200z^2 \right),$$

where $x = \Theta^{\#}(2)$, $y = \Theta^{\#}(4)$, $z = \Theta^{\#}(6)$.

References

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- [2] C. Elsner, S. Shimomura, and I. Shiokawa, “Algebraic relations for reciprocal sums of Fibonacci numbers,” *Preprint* (2006).
- [3] Nesterenko, Yu. V.: Modular functions and transcendence questions. *Mat. Sb.* **187**, 65–96 (1996); English transl. *Sb. Math.* **187**, 1319–1348 (1996)